Bound State Solutions of Klein-Gordon Equation with Manning-Rosen Plus a Class of Yukawa Potential Using Pekeris-Like Approximation of the Coulomb Term and Parametric Nikiforov-Uvarov

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Abstract

The solutions of the Klein-Gordon equation with Manning-Rosen plus a class of Yukawa potential (MRCYP) have been presented using the Pekeris-like approximation and parametric Nikiforov-Uvarov (NU) method. The bound state energy eigenvalues and the corresponding un-normalized eigen functions are obtained in terms of Jacobi polynomials. Additionally, inversely quadratic Yukawa Manning-Rosen and coulomb potentials have been recovered from the mixed potential.

Keywords: Klein-Gordon equation, Manning-Rosen potential, Yukawa potential, Pekeris, ParametricNikiforov-Uvarov method, Jacobi polynomials

Introduction

There has been a growing interest in investigating the approximate solutions of the Klein-Gordon equation and relativistic wave equations for some physical potential models. This is due to the fact that the analytical solutions contain all the necessary information for the quantum system under consideration [1]. Taking the relativistic effects into account, a quantum system in a potential field should be described with the Klein–Gordon equation and Dirac equation. When a quantum system is in a strong potential field, the relativistic effect must be considered, which gives the correction for nonrelativistic quantum mechanics. Over the years, different researchers have investigated the bound state solutions of Klein–Gordon equations in some typical potential fields, such as Coulomb potential [2], Poschl–Teller potential [3], Rosen–Morse potential [4], Eckart potential [5], noncentral potential [6], rotating Morse potential [7], ring-shaped potential [8], Hartmann potential [9], double ring-shaped oscillator potential [10], ring-shaped Kratzer potential [11], and double ring-shaped Kratzer potential [12]. Recently, our research group has also reported the analytical solutions to the Klein-Gordon equation with different mixed potentials such as generalized wood-saxon plus Mie-type potential (GWSMP) [13], modified echart plus inverse square molecular potential (MEISMP) [14].

The purpose of this paper is to solve the Klein-Gordon equation for a novel mixed type potential consisting of Mannin-Rosen and a class of Yukawa-like potential(MRCYP) using the parametric NU method.

Nikiforov-Uvarov (Nu) Method

NU method is based on the solutions of a generalized second-order linear differential equation with special orthogonal functions. The Klein-Gordon equationof the type as:

$$\left[-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+(S(r)+M)^2\right]\psi(r,\theta,\phi)=0, \tag{1}$$

can be solved by this method. To do this,equation (1) is transformed into equation of hypergeometric type with appropriate coordinate transformation $s=s(r)$ to get

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{2}$$

To solve equation (2) we can use the parametric NU method. The parametric generalization of the NU method is expressed by the generalized hypergeometrictype equation

$$\psi''(s)+\frac{(c_1-c_2 s)}{s(1-c_3 s)}\psi'(s)+\frac{1}{s^2(1-c_3 s)^2}\left[-\epsilon_1 s^2+\epsilon_2 s-\epsilon_3\right]\psi(s)=0, \tag{3}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials atmost second degree, and $\tilde{\tau}(s)$ is a first degree polynomial.

The eigenfunctions (Equation 4) and corresponding eigenvalues (Equation 5) to the equation become

$$\psi(s)=N_n s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s), \tag{4}$$

$$(c_2-c_3)n+c_3 n^2-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+c_7+2c_3 c_8+2\sqrt{c_8 c_9}=0, \tag{5}$$

Where

$$c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\epsilon_1,\ c_7=2c_4c_5-\epsilon_2,\ c_8=c_4^2+\epsilon_3,\ \ c_9=c_3c_7+c_3^2c_8+c_6,\ \ c_{10}=c_1+2c_4+2\sqrt{c_8},\ c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right),\ c_{12}=c_4+\sqrt{c_8},$$
$$c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right). \tag{6}$$

N_n is the normalization constant and $P_n^{(\alpha,\beta)}$are the Jacobi polynomials.

SOLUTIONS OF THE RADIAL PART OF KLEIN-GORDON EQUATION WITH MRCYP POTENTIAL:

The radial part of the Klein-Gordon Equation with vector V(r) potential = scalar S(r) potentia tomic units ($\hbar=c=1$) is given as

$$\frac{d^2R(r)}{dr^2}+\left[(E^2-M^2)-2(E+M)V(r)\right]R(r)=0\ , \tag{7}$$

Where $\lambda=l(l+1)$ and $V(r)$ is the potential energy function. The Manning-Rosen potential (MRP) is given as [14]

$$V(r)=-\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] \tag{8}$$

In Equation (8),C and D are constants.

The Class ofYukawa potential (CYP) is given as [15]

$$V(r)=-\frac{V_0e^{-\alpha r}}{r}-\frac{V_0'e^{-2\alpha r}}{r^2}, \tag{9}$$

Where $V_0$and $V_0'$are the potential depth of the CYP and α is an adjustable positive parameter.

The sum of these potentials known as MRCYP is given as

$$V(r) = -\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - \frac{V_0 e^{-\alpha r}}{r} - \frac{V_0' e^{-2\alpha r}}{r^2} \quad (10)$$

Making the transformation $s = e^{-\alpha r}$ Equation (10) becomes

$$V(s) = -\left[\frac{CS+DS^2}{(1-S)^2}\right] - \frac{V_0 \alpha S}{1-S} - \frac{V_0' \alpha^2 S^2}{(1-S)^2} \quad (11)$$

Again, applying the transformation $s = e^{-\alpha r}$ to get the form that NU method is applicable, equation (7) gives a generalized hypergeometric-type equation as

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{(1-s)s}\frac{dR(s)}{ds} + \frac{1}{(1-s)^2 s^2}[-(\beta^2 - F + B - G)s^2 + (2\beta^2 + A + B)s - (\beta^2)]R(s) = 0, (12)$$

Where

$$\lambda = 0, -\beta^2 = \frac{E^2-M^2}{4\alpha^2}, A = 2\left(\frac{E+M}{\alpha^2}\right)C, B = 2\left(\frac{E+M}{\alpha}\right)V_0, F = 2\left(\frac{E+M}{\alpha^2}\right)D, G = 2(E+M)V_0', \frac{1}{r} \approx \frac{\alpha}{(1-e^{-\alpha r})} \approx \frac{\alpha}{(1-S)} \quad (13)$$

Comparing Equation (12) with Equation (3) yields the following parameters

$$c_1 = c_2 = c_3 = 1, c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \beta^2 + B - F - G, c_7 = -2\beta^2 - A - B, c_8 = \beta^2, c_9 = \frac{1}{4} - (A+F+G), c_{10} = 1 + 2\sqrt{\beta^2}, c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} - A - F - G} + \sqrt{\beta^2}\right), c_{12} = \sqrt{\beta^2}, c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} - A - F - G} + \sqrt{\beta^2}\right), \epsilon_1 = \beta^2 + B - F - G, \epsilon_2 = 2\beta^2 + A + B, \epsilon_3 = \beta^2, \quad (14)$$

Now using Equations (5), (13) and (14) we obtain the energy eigen spectrum of the MRYP as

$$\beta^2 = \left[\frac{A+B-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-A-F-G}}{(2n+1)+2\sqrt{\frac{1}{4}-A-F-G}}\right]^2 \quad (15)$$

Equation (15) can be solved explicitly and the energy eigen spectrum of MRYP becomes

$$E^2 - M^2 = -4\alpha^2\left\{\left[\frac{2\left(\frac{E+M}{\alpha^2}\right)C + 2\left(\frac{E+M}{\alpha}\right)V_0 - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D - 2(E+M)V_0'}}{(2n+1) + 2\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D - 2(E+M)V_0'}}\right]^2\right\} (16)$$

We now calculate the radial wave function of the MRCYP as follows:

The weight function $\rho(s)$ is given as [19]

$$\rho(s) = s^{c_{10}-1}(1 - c_3 s)^{\frac{c_{11}}{c_3} - c_{10} - 1} \quad (17)$$

Using Equation (14) we get the weight function as

$$\rho(s) = s^U (1-s)^V, \quad (18)$$

where $U = 2\sqrt{\beta^2}$ and $V = 2\sqrt{\frac{1}{4} - A - F - G}$

Also we obtain the wave function $\chi(s)$ as [19]

$$\chi(s) = P_n^{c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1}(1 - 2c_3 s), \quad (19)$$

Using Equation (14) we get the function $\chi(s)$ as

$$\chi(s) = P_n^{(U,V)}(1-2s), \quad (20)$$

where $P_n^{(U,V)}$ are Jacobi polynomials

Lastly,

$$\varphi(s) = s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}, \quad (21)$$

and using equation (14) we get

$$\varphi(s) = s^{U/2}(1-s)^{V-1/2} \quad (22)$$

We can then obtain the radial wave function

$$R_n(s) = N_n \varphi(s) \chi_n(s), \quad (23)$$

As

$$R_n(s) = N_n s^{U/2}(1-s)^{(V-1)/2} P_n^{(U,V)}(1-2s), \quad (24)$$

where n is a positive integer and N_n is the normalization constant.

4. DISCUSSION

We have solved the radial Schrödinger equation and obtained the energy eigen values for the Manning-Rosen plus Class Yukawa potential (MRCYP) in Equation (16).

The following cases are considered:

Case 1: If $C = D = V_0{}' = 0$ in Equation (10), the potential turns back into the Yukawa potential and Equation (16) yields the energy eigen values of the Yukawa potential as

$$E^2 - M^2 = -4\alpha^2 \left[\frac{2\left(\frac{E+M}{\alpha}\right)V_0 - (n+1)^2}{2(n+1)}\right]^2 \quad (25)$$

Equation (25) is similar to that which was reported by Jia*et al.*[17]

Case 2: If $\alpha \to 0$ in Equation (25), the energy eigen values for Coulomb potential becomes

$$E^2 - M^2 = -4\frac{(E+M)}{(n+1)^2} \quad (26)$$

Case 3: If $V_0 = V_0{}' = 0$ the potential in Equation (10) yields the Manning-Rosen potential with energy eigen values given as

$$E^2 - M^2 = -4\alpha^2 \left[\frac{2\left(\frac{E+M}{\alpha^2}\right)C - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D}}{(2n+1) + 2\sqrt{\frac{1}{4} - 2\left(\frac{E+M}{\alpha^2}\right)C - 2\left(\frac{E+M}{\alpha^2}\right)D}}\right]^2 \quad (27)$$

Eq. (27) is similar to the equation reported in ref. [16] for Manning-Rosen potential.

Case 4: If $C = D = V_0 = 0$; the potential in Equation 10 yields the Inversely quadratic Yukawa potential with energy eigen value given as

$$E^2 - M^2 = -4\alpha^2 \left\{\left[\frac{-\left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - 2(E+M)V_0{}'}}{(2n+1) + 2\sqrt{\frac{1}{4} - 2(E+M)V_0{}'}}\right]^2\right\} \quad (28)$$

Case 5: If C = D = 0; the potential in Equation 10 yields a class of the Yukawa potential with energy eigenvalue given as

$$E^2 - M^2 = -4\propto^2 \left\{\left[\frac{2\left(\frac{E+M}{\alpha}\right)V_0 - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} - 2(E+M)V_0{}'}}{(2n+1) + 2\sqrt{\frac{1}{4} - 2(E+M)V_0{}'}}\right]^2\right\} \tag{29}$$

Conclusion

Analytical solutions of Klein–Gordon equation in case of Manning-Rosen plus a class of Yukawa potentials have been obtained using parametric form of NU method. With a good approximation to centrifugal term, we have obtained energy eigenvalues and un-normalized wave function in terms of Jacobi polynomials. Special cases for the potentials are discussed indicating usefulness for other physical systems.

References

[1] R.C. Wang, C.Y. Wong, Phys. Rev. D 38 (1988) 348.
[2] A.D. Alhaidari, H. Bahlouli, A. Al-Hasan, Phys. Lett. A 349 (2006) 87.
[3] G. Kocak, F. Ta, skın, Ann. Phys. (Berlin) 522 (2010) 802.
[4] L.Z. Yi, Y.F. Diao, J.Y. Liu, C.S. Jia, Phys. Lett. A 333 (2004) 212.
[5] G.F. Wei, S.H. Dong, V.B. Bezerra, Int. J. Mod. Phys. A 24 (2009) 161.
[6] F. Yasuk, A. Durmus, I. Boztosun, J. Math. Phys. 47 (2006) 082302.
[7] O. Bayrak, A. Soylu, I. Boztosun, J. Math. Phys. 51 (2010) 112301.
[8] B.J. Falaye, J. Math. Phys. 53 (2012) 082107.
[9] C.Y. Chen, F.L. Lu, D.S. Sun, Phys. Scr. 78 (2008) 015014.
[10] F. Yasuk, A. Durmus, Phys. Scr. 77 (2008) 015005
[11] S.M. Ikhdair, R. Sever, Cent. Eur. J. Phys. 6 (2008) 141.
[12] A. Durmus, F. Yasuk, J. Chem. Phys. 126 (2007) 074108
[13] B.I. Ita, B. E. Nyong., H. Louis, T.O. Magu (2017):*J. Chem. Soc. Nigeria, Vol. 41, No. 2, pp21-26*
[14] B.I. Ita, B.E. Nyong, N.O. Alobi, H. Louis and T.O. Magu *Equatorial Journal of Computational and Theoretical Sciemces*, Vol.1, Issue 1, (2016), pp. 55-64.
[15] B.I. Ita, H. Louis, T.O. Magu and N.A Nzeata-Ibe *World scientific news*, WSN 70(2) (2017) 312-319
[16] C. Jia, T. Chenb, S. Hec Physics Letters A 377 (2013) 682–686
[17] A D Antia1, A N Ikot1, H Hassanabadiand E MaghsoodiIndian J Phys DOI 10.1007/s12648-013-0336-y