Analytic Spin and Pseudospin Solutions to the Dirac Equation for the Manning-Rosen Plus Eckart Potential and Yukawa-Like Tensor Interaction

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Abstract

We solve the Dirac equation for the Manning-Rosen plus Eckart potential including a Coulomb-like tensor potential with arbitrary spin–orbit coupling quantum number κ. In the framework of the spin and pseudospin (pspin) symmetry, we obtain the energy eigenvalue equation and the corresponding eigenfunctions in closed form by using the Nikiforov–Uvarov method. Also special cases of the potential have been considered and their energy eigen values as well as their corresponding eigen functions have been obtained for both relativistic and non-relativistic scopes.

Keywords: Dirac equation, Manning-Rosen potential, Eckart potential, spin and pseudospin symmetry, Nikiforov-Uvarov Method.

Introduction

By taking into account the relativistic effects due to the speed and spin of the particles, relativistic equations like Dirac and Klein–Gordon must be considered, but these equations can only be solved analytically for only few special interactions, or by imposing some conditions on the potentials to obtain the analytic solution [1, 2]. In numerous physics applications such as in the areas of nuclear physics and high-energy physics, one of the most significant problems is how to obtain exact analytical solutions of the relativistic equations like the Dirac and Klein–Gordon equations for mixed vector and scalar potentials [3]. The Dirac and Klein–Gordon wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics. In recent years, a lot of effort has been put into solving these relativistic wave equations for various potentials by using different methods [4, 5]. Some researchers have investigated the Dirac equation by using a variety of potentials and different methods, such as the spin symmetry in the antinucleon spectrum and tensor type Coulomb potential with spin–orbit number k in a state of spin symmetry and p-spin symmetry [6],bound states of the Dirac equation with position-dependent mass for the Eckart potential [7], the exact solution of Klein–Gordon with the Poschl–Teller double-ring-shaped Coulomb potential [8],the exact solution of the Dirac equation for the Coulomb potential plus NAD potential by using the Nikorov–Uvarov method [9], the potential Deng–Fan and the Coulomb potential tensor using the asymptotic iteration method (AIM) [10],the potential Poschl–Teller plus the Manning Rosen radial section with the hypergeometry method [11], the solution ofKlein–Gordon equation for Hulthen non-central potential inradial part with Romanovski polynomial[12],and the solution of the Schrodinger equation with the Hulthen plus Manning–Rosen potential [13], the Scarf potential with the new tensor coupling potential for spin and pseudospin symmetries using Romanovski polynomials [14],for the q-deformed hyperbolic Poschl–Teller potential and the trigonometric Scarf II noncentral potential by using AIM [15],eigensolutions of the deformed Woods–Saxon potential via AIM [16],approximate solutions of the Klein Gordon equation with an improved Manning Rosen potential in D-dimensions using SUSYQM [17], and eigen spectra of the Dirac equation for a deformed Woods–Saxon potential via the similarity transformation [18].

The conventional NU method, which received much interest, has been introduced for solving the Schrödinger, Dirac and Klein–Gordon wave equations. This method is applicable to different quantum mechanical systems. The discussion of the relativistic behaviour of spin-1/2 particles requires an understanding of the single-particle spectrum and the exact solutions of the Dirac equation with the vector and scalar potentials.

In this work,our aim is to solve the Dirac equation for the Manning-Rosen plus Eckart (MRE) potential in the presence of spin and pspin symmetries and by includinga Yukawa-like tensor potential. . The MRCY potential takes the following form:

$$V(r) = -\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - \frac{V_0}{r} - \frac{V_1 e^{-\alpha r}}{r} - \frac{V_2 e^{-2\alpha r}}{r^2} \qquad (1)$$

where α is the screening parameter, C, D and A, B are depths of the potential.

The Dirac equation with tensor coupling potential

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential $S(r)$, a repulsivevectorpotential $V(r)$ andatensorpotential$U(r)$ (in unitsħ = c = 1) is

$$[\vec{\alpha}\cdot\vec{p} + \beta(M + S(r)) - i\beta\vec{\alpha}\cdot\vec{r}U(r)]\psi(\vec{r}) = [E - V(r)]\psi(\vec{r}). \qquad (2)$$

where E is the relativistic binding energy of the system ,$p = -i\vec{\nabla}$ isthethree-dimensionalmomentumoperatorand M is the mass of the fermionic particle. $\vec{\alpha}$ and β are the 4×4usual Dirac matrices given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad (3)$$

where I isthe2×2unitarymatrixand $\vec{\sigma}$ arethree-vectorspin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4)$$

The eigenvalues of the spin–orbit coupling operator are$\kappa = \left(j+\frac{1}{2}\right) > 0$ and $\kappa = -\left(j+\frac{1}{2}\right) < 0$ for unaligned spin $j = l - \frac{1}{2}$ and aligned spin $j = l + \frac{1}{2}$, respectively. The set(H^2, K, J^2, J_z)can be taken as the complete set of conservative quantities with$\vec{J}$ being the total angular momentum operator and $K = (\vec{\sigma}.\vec{L} + 1)$is the spin–orbit where $\vec{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n,\kappa}(\vec{r}) = \begin{pmatrix} f_{n,\kappa}(\vec{r}) \\ g_{n,\kappa}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n,\kappa}(r) & Y_{jm}^{l}(\theta,\varphi) \\ iG_{n,\kappa}(r) & Y_{jm}^{\tilde{l}}(\theta,\varphi) \end{pmatrix}, \tag{5}$$

where $f_{n,\kappa}(\vec{r})$ is the upper (large) component and $g_{n,\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y_{jm}^{l}(\theta,\varphi)$ and $Y_{jm}^{\tilde{l}}(\theta,\varphi)$ are spin and pspin spherical harmonics, respectively, and m is the projection of the angular momentum on the $z-axis$. Substituting Equation (5) into Equation (2) and making use of the following relations

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right), \tag{6a}$$

$$\left(\vec{\sigma}\cdot\vec{P}\right) = \vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{P} + i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{6b}$$

together with the following properties

$$\begin{aligned} \left(\vec{\sigma}\cdot\vec{L}\right)Y_{jm}^{\tilde{l}}(\theta,\varphi) &= (\kappa-1)Y_{jm}^{\tilde{l}}(\theta,\varphi), \\ \left(\vec{\sigma}\cdot\vec{L}\right)Y_{jm}^{l}(\theta,\varphi) &= -(\kappa-1)Y_{jm}^{l}(\theta,\varphi), \\ (\vec{\sigma}\cdot\hat{r})Y_{jm}^{\tilde{l}}(\theta,\varphi) &= -Y_{jm}^{l}(\theta,\varphi), \\ (\vec{\sigma}\cdot\hat{r})Y_{jm}^{l}(\theta,\varphi) &= -Y_{jm}^{\tilde{l}}(\theta,\varphi), \end{aligned} \tag{7}$$

one obtains two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ as

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) = (M + E_{n\kappa} - \Delta(r))G_{n,\kappa}(r), \tag{8a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n,\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))F_{n,\kappa}(r), \tag{8b}$$

where

$$\Delta(r) = V(r) - S(r), \tag{9a}$$

$$\Sigma(r) = V(r) + S(r), \tag{9b}$$

After eliminating $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ in Equations (8), we obtain the following two Schrodinger-like differential equations for the upper and lower radial spinor components:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r)\right]F_{n,\kappa}(r) + \frac{\frac{d\Delta(r)}{dr}}{M + E_{n\kappa} - \Delta(r)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r)$$
$$=[(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]F_{n,\kappa}(r) \tag{10}$$

$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)+\frac{dU(r)}{dr}-U^2(r)\right]G_{n,\kappa}(r)+\frac{\frac{d\Sigma(r)}{dr}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n,\kappa}(r)=[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))]G_{n,\kappa}(r)$,(11)

respectively, where $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$ and $\kappa(\kappa+1)=l(l+1)$.
The quantum number κ is related to the quantum numbers for spin symmetry l and pspin symmetry $\tilde{l}$ as

$$\kappa=\begin{cases}-(l+1)=-\left(j+\frac{1}{2}\right)(s_{1/2},p_{3/2},etc)\\ j=l+\frac{1}{2},\ aligned\ spin\ (\kappa<0),\\ +l=+\left(j+\frac{1}{2}\right)(p_{1/2},d_{3/2},etc)\\ j=l-\frac{1}{2},\ unaligned\ spin\ (\kappa>0),\end{cases} \qquad (12)$$

and the quasidegenerate doublet structure can be expressed in terms of a pspin angular momentum $\hat{s}=1/2$ and pseudo-orbital angular momentum $\hat{l}$, which is defined as

$$\kappa=\begin{cases}-\hat{l}=-\left(j+\frac{1}{2}\right)(s_{1/2},p_{3/2},etc)\\ j=\hat{l}-\frac{1}{2},\ aligned\ spin\ (\kappa<0),\\ +(\hat{l}+1)=+\left(j+\frac{1}{2}\right)(d_{3/2},f_{5/2},etc)\\ j=\hat{l}+\frac{1}{2},\ unaligned\ spin\ (\kappa>0),\end{cases} \qquad (13)$$

where $\kappa=\pm1,\pm2,\ldots$. For example, $(1s_{1/2},\ 0d_{3/2})$ and $(0p_{3/2},\ 0f_{5/2})$ can be considered as pspin doublets

Spin symmetry limit

In the spin symmetry limit, $\frac{d\Delta(r)}{dr}=0$ or $\Delta(r)=C_s$ =constant, with $\Sigma(r)$ taking as the MRE potential Equation (1) and the coulomb-like tensor potential. i.e

$$\Sigma(r)=V(r)=-\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]-A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})}+B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}, \qquad (14)$$

$$U(r)=-\frac{H}{r}e^{-\alpha r}, \qquad (15)$$

Underthissymmetry, Equation(10)isrecast in the simpleform

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-\frac{2\kappa He^{-\alpha r}}{r^2}-\frac{He^{-\alpha r}}{r^2}-\frac{\alpha He^{-\alpha r}}{r}-\frac{H^2e^{-2\alpha r}}{r^2}\right]F_{n,\kappa}(r)=\left[\gamma\left(-\left[\frac{Ce^{-\alpha r}+De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]-A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})}+B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right)+\beta^2\right]F_{n,\kappa}(r) \qquad (16a)$$

where $\kappa=l$ and $\kappa=-l-1$ for $\kappa<0$ and $\kappa>0$, respectively. Also, $\gamma=(M+E_{n\kappa}-C_s)$ and $\beta^2=(M-E_{n\kappa})(M+E_{n\kappa}-C_s)$. (16b)

Pseudospin symmetry limit

It has been shownthatthereisaconnectionbetween pspin symmetryandnearequalityofthetimecomponentofavector potential and the scalar potential [19], $V(r) \approx -S(r)$. After that, it was derived that if $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{ps}$ =constant, then pspin symmetry is exact in the Dirac equation. Here, we are taking $\Delta(r)$ as the MRE potential Equation(1) and the tensor potential as the Coulomb-like potential. Thus, Equation(11)isrecast in the simple form

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - \frac{2\kappa H e^{-\alpha r}}{r^2} + \frac{He^{-\alpha r}}{r^2} + \frac{\alpha H e^{-\alpha r}}{r} - \frac{H^2 e^{-2\alpha r}}{r^2}\right] G_{n,\kappa}(r) = \left[\tilde{\gamma}\left(-\left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - A\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + B\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}\right) + \tilde{\beta}^2\right] G_{n,\kappa}(r) \quad (17a)$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Also, $\tilde{\gamma} = (E_{n\kappa} - M - C_{ps})$ and

$$\tilde{\beta}^2 = (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps}) \,. \quad (17b)$$

to obtain the analytic solution, we use an approximation for the centrifugal term as []

$$\frac{1}{r^2} = \frac{\alpha^2}{(1-e^{-\alpha r})^2} \quad (18)$$

Finally, for the solutions to Equations (16) and (17) with the above approximation, we will employ the NU method, which is briefly introduced in the following section

The Nikiforov–Uvarov (NU) method

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi_n(s) = 0 \quad (19)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is first degree polynomials.The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}\left[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3\right]\Psi(s) = 0 \quad (20)$$

Thus eqn. (2) can be solved by comparing it with equation (3) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s), \sigma(s) = s(1 - c_3 s), \tilde{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \quad (21)$$

The parameters obtainable from equation (4) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equation respectively

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \quad (22)$$

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \quad (23)$$

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While the wave function is given as

$$\Psi_n(s) = N_{n,l} S^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s) \qquad (24)$$

Where

$$c_4 = \frac{1}{2}(1 - c_1),\ c_5 = \frac{1}{2}(c_2 - 2c_3),\ c_6 = c_5{}^2 + \epsilon_1,\ c_7 = 2c_4 c_5 - \epsilon_2,\ c_8 = c_4{}^2 + \epsilon_3,$$
$$c_9 = c_3 c_7 + c_3{}^2 c_8 + c_6,\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},\ c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c3\sqrt{c_8}\right)$$
$$c_{12} = c_4 + \sqrt{c_8},\ c_{13} = c_5 - \left(\sqrt{c_9} + c3\sqrt{c_8}\right) \qquad (25)$$

and P_n is the orthogonal polynomials.

Solutions to the Dirac equation

We will now solve the Dirac equation with the MRE potential and tensor potential by using the NU method.

The spin symmetric case

To obtain the solution to Equation (16), by using the transformation $s = e^{-\alpha r}$, we rewrite it as follows:

$$\frac{d^2 F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2}\Big[-\kappa(\kappa+1) - 2\kappa H s - 2Hs + Hs^2 - H^2 s^2 + \frac{\gamma}{\alpha^2}(Cs + Ds^2 + As(1-s) + Bs) - \frac{\beta^2}{\alpha^2}(1-s)^2\Big] F_{n,\kappa}(s) = 0, \qquad (26)$$

Eq. (26) is further simplified as

$$\frac{d^2 F_{n,\kappa}(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-\left(\frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}D + \frac{\gamma}{\alpha^2}A + H^2 - H\right)s^2 + \left(\frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}B + \frac{\gamma}{\alpha^2}A + \frac{\gamma}{\alpha^2}C - 2\kappa H - 2H\right)s - \left(\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)\right)\right] F_{n,\kappa}(s) = 0, \qquad (27)$$

Comparing eq. (27) with eq. (20), we obtain

$$c_1 = 1,\quad \epsilon_1 = \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}D + \frac{\gamma}{\alpha^2}A - \gamma V_2 + H^2 - H$$
$$c_2 = 1,\quad \epsilon_2 = \frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}B + \frac{\gamma}{\alpha^2}A + \frac{\gamma}{\alpha^2}C - 2\kappa H - 2H \qquad (28)$$
$$c_3 = 1,\quad \epsilon_3 = \frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)$$

and from Eq. (25), we further obtain

$$c_4 = 0, \qquad c_5 = -\frac{1}{2},$$
$$c_6 = \frac{1}{4} + \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}D + \frac{\gamma}{\alpha^2}A + H^2 - H, \qquad c_7 = -\left(\frac{2\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}B + \frac{\gamma}{\alpha^2}A + \frac{\gamma}{\alpha^2}C - 2\kappa H - 2H\right),$$
$$c_8 = \frac{\beta^2}{\alpha^2} + \kappa(\kappa+1), \qquad c_9 = \left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2}D - \frac{\gamma}{\alpha^2}C + \frac{\gamma}{\alpha^2}B, \text{ where } \eta_\kappa = \kappa + H + 1,$$

$$c_{10} = 1 + 2\sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)},$$

$$c_{11} = 2 + 2\left(\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2}D - \frac{\gamma}{\alpha^2}C + \frac{\gamma}{\alpha^2}B} + \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}\right), \quad (29)$$

$$c_{12} = \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)},$$

$$c_{13} = -\frac{1}{2} - \left(\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2}D - \frac{\gamma}{\alpha^2}C + \frac{\gamma}{\alpha^2}B} + \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}\right)$$

In addition, the energy eigenvalue equation can be obtained by using Eq. (23) as follows:

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2}D - \frac{\gamma}{\alpha^2}C + \frac{\gamma}{\alpha^2}B} + \sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}\right)^2 = \frac{\beta^2}{\alpha^2} - \frac{\gamma}{\alpha^2}D + \frac{\gamma}{\alpha^2}A + H^2 - H \quad (30)$$

By substituting the explicit forms of γ *and* β^2 after Equation (16) into Equation (30), one can readily obtain the closed form for the energy formula.

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{D}{\alpha^2}(M + E_{n\kappa} - C_s) - \frac{C}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{B}{\alpha^2}(M + E_{n\kappa} - C_s)} + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa} - C_s)\right) + \kappa(\kappa+1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa} - C_s)\right) - \frac{D}{\alpha^2}(M + E_{n\kappa} - C_s) + \frac{A}{\alpha^2}(M + E_{n\kappa} - C_s) + H^2 - H \quad (31)$$

On the other hand, to find the corresponding wave functions, referring to equation (29) and eq. (24), we obtain the upper component of the Dirac spinor from eq. 24 as

$$F_{n,\kappa}(s) = B_{n,\kappa} s^{\sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)}} (1-s)^{\frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2}D - \frac{\gamma}{\alpha^2}C + \frac{\gamma}{\alpha^2}B}} P_n^{\left(2\sqrt{\frac{\beta^2}{\alpha^2} + \kappa(\kappa+1)},\, 2\sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{\gamma}{\alpha^2}D - \frac{\gamma}{\alpha^2}C + \frac{\gamma}{\alpha^2}B}\right)}(1 - 2s) \quad (32)$$

where $B_{n,\kappa}$ is the normalization constant. The lower component of the Dirac spinor can be calculated from equation (8a)

$$G_{n,\kappa}(r) = \frac{1}{(M + E_{n\kappa} - C_s)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) \quad (33)$$

where $E_{n\kappa} \neq -M + C_s$.

The pseudospin symmetric case

To avoid repetition in the solution of Equation (17), we follow thesameproceduresexplainedinsection4.1andhenceobtain the following energy eigenvalue equation:

$$\left(n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 - \frac{\bar{\gamma}}{\alpha^2}D - \frac{\bar{\gamma}}{\alpha^2}C + \frac{\bar{\gamma}}{\alpha^2}B} + \sqrt{\frac{\bar{\beta}^2}{\alpha^2} + \kappa(\kappa-1)}\right)^2 = \frac{\bar{\beta}^2}{\alpha^2} - \frac{\bar{\gamma}}{\alpha^2}D + \frac{\bar{\gamma}}{\alpha^2}A + H^2 + H$$

and the corresponding wave functions for the upper Dirac spinor as

$$G_{n,\kappa}(r) = \tilde{B}_{n,\kappa} s^{\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\kappa(\kappa-1)}}(1-s)^{\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{\tilde{\gamma}}{\alpha^2}D-\frac{\tilde{\gamma}}{\alpha^2}C+\frac{\tilde{\gamma}}{\alpha^2}B}} P_n^{\left(2\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\kappa(\kappa-1)},\,2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{\tilde{\gamma}}{\alpha^2}D-\frac{\tilde{\gamma}}{\alpha^2}C+\frac{\tilde{\gamma}}{\alpha^2}B}\right)}(1-2s) \quad (35)$$

were $\Lambda_\kappa = \kappa + H$ and $\tilde{B}_{n,\kappa}$ is the normalization constant. Finally,the Upper-spinor component oftheDiracequationcan be obtained via equation (8b) as

$$F_{n,\kappa}(r) = \frac{1}{(M - E_{n\kappa} + C_{ps})}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right) G_{n,\kappa}(r) \quad (36)$$

where $E_{n\kappa} \neq M + C_{ps}$.

Discussions

In this section, we are going to study some special cases of the energy eigenvalues given by Eqs. (31) and (35) for the spin and pseudospin symmetries, respectively.

Case 1. If one sets $C_s = 0$, $C_{ps} = 0$, $A = B = 0$ in Eq. (31) and Eq. (35), we obtain the energy equation of Manning-Rosen potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 - \frac{D}{\alpha^2}(M + E_{n\kappa}) - \frac{C}{\alpha^2}(M + E_{n\kappa})} + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \kappa(\kappa+1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) - \frac{D}{\alpha^2}(M + E_{n\kappa}) + H^2 - H \quad (38)$$

and

$$\left(n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 - \frac{D}{\alpha^2}(E_{n\kappa} - M) - \frac{C}{\alpha^2}(E_{n\kappa} - M)} + \sqrt{\frac{1}{\alpha^2}(M + E_{n\kappa})(M - E_{n\kappa}) + \kappa(\kappa-1)}\right)^2 = \frac{1}{\alpha^2}(M + E_{n\kappa})(M - E_{n\kappa}) - \frac{D}{\alpha^2}(E_{n\kappa} - M) + H^2 + H \quad (39)$$

Case 2: If one sets $C_s = 0$, $C_{ps} = 0$, $C = D = 0$ in Eq. (31) and Eq. (35), we obtain the energy equation of Eckart potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n + \frac{1}{2} + \sqrt{\left(\eta_\kappa - \frac{1}{2}\right)^2 + \frac{B}{\alpha^2}(M + E_{n\kappa})} + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \kappa(\kappa+1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \frac{A}{\alpha^2}(M + E_{n\kappa}) + H^2 - H \quad (40)$$

and

$$\left(n + \frac{1}{2} + \sqrt{\left(\Lambda_\kappa - \frac{1}{2}\right)^2 + \frac{B}{\alpha^2}(E_{n\kappa} - M)} + \sqrt{\frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \kappa(\kappa-1)}\right)^2 = \frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \frac{A}{\alpha^2}(E_{n\kappa} - M) + H^2 + H \quad (41)$$

Case 3: If one sets $C_s = 0$, $C_{ps} = 0, B = 0$, $C = 0, D = 0$, in Eq. (31) and Eq. (35), we obtain the energy equation of Hulthen potential for spin and pseudospin symmetric Dirac theory respectively,

$$\left(n + \eta_\kappa + \sqrt{\frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \kappa(\kappa + 1)}\right)^2 = \frac{1}{\alpha^2}\left((M - E_{n\kappa})(M + E_{n\kappa})\right) + \frac{A}{\alpha^2}(M + E_{n\kappa}) + H^2 - H \quad (42)$$

and

$$\left(n + \Lambda_\kappa + \sqrt{\frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right) + \kappa(\kappa - 1)}\right)^2 = \frac{1}{\alpha^2}\left((M + E_{n\kappa})(M - E_{n\kappa})\right)\frac{A}{\alpha^2}(M + E_{n\kappa}) + H^2 + H \quad (43)$$

Case 6: Let us now discuss the relativistic limit of the energy eigenvalues and wavefunctions of our solutions. If we take $C_s = 0, H = 0$ and put $S(r) = V(r) = \Sigma(r)$, the nonrelativistic limit of energy Equation (31) for MRE potential and wave function 32 under the following appropriate transformations $M + E_{n\kappa} \to \frac{2\mu}{\hbar^2}$, and $M - E_{n\kappa} \to -E_{nl}$ becomes

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1) - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu A}{\alpha^2\hbar^2} + \frac{2\mu B}{\alpha^2\hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2} + \frac{2\mu B}{\alpha^2\hbar^2}}}{(2n+1) + 2\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2} + \frac{2\mu B}{\alpha^2\hbar^2}}}\right]^2 - l(l+1)\right\}$$

and the associated wave functions $F_{n\kappa}(s) \to R_{n,l}(s)$ are

$$R_{n,l}(s) = N_{n,l}s^{U/2}(1-s)^{(V-1)/2}P_n^{(U,V)}(1-2s), \quad (44)$$

where $U = 2\sqrt{\frac{2\mu E_{nl}}{\alpha^2\hbar^2} + l(l+1)}$ and $V = 2\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2} + \frac{2\mu B}{\alpha^2\hbar^2}}$ (45)

Case 7: If $A = B = 0$ in Eq. (44), we obtain the energy equation of Manning-Rosen potential in the non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1) - \frac{2\mu C}{\alpha^2\hbar^2} + \left(n^2 + n + \frac{1}{2}\right) + (2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2}}}{(2n+1) + 2\sqrt{\left(l+\frac{1}{2}\right)^2 - \frac{2\mu C}{\alpha^2\hbar^2} - \frac{2\mu D}{\alpha^2\hbar^2}}}\right]^2 - l(l+1)\right\} \quad (46)$$

Case 8: If $C = D = 0$ in Eq. (44), we obtain the energy equation of the Eckart potential inthe non-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu A}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu B}{\alpha^2\hbar^2}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu B}{\alpha^2\hbar^2}}}\right]^2 - l(l+1)\right\} \quad (47)$$

Case 9: If $B = 0, C = D = 0$ in eq. (47), we obtain the energy equation of the Hulthen potential in thenon-relativistic limit

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu A}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2}}\right]^2 - l(l+1)\right\} \quad (48)$$

Conclusion

We have studied Analytic spin and pseudospin solutions to the Dirac equation for the Manning-Rosen plus Eckart potential and Yukawa-like tensor interaction. We have obtained the energy eigenvalue equations and the related two-component spinor wave functions with the help of Nikiforov–Uvarov method.

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